TULLIO REGGE
An Eclectic Genius
From Quantum Gravity to Computer Play

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Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Control Number: 2019044511

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

TULLIO REGGE: AN ECLECTIC GENIUS
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ISBN 978-981-121-343-4

For any available supplementary material, please visit
https://www.worldscientific.com/worldscibooks/10.1142/11643#t=suppl

To Rosanna Cester

Contents

PART I

Black Hole Stability under Pertubations

The Ponzano–Regge Model

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We present an overview of the role of the (Racah–)Wigner $6j$ symbol as the basic building block underlying such different fields as state sum models for quantum geometry, topological quantum field theory, statistical lattice models and quantum computing. Focusing on the geometric side, the results found in the seminal paper *Semiclassical Limit of Racah Coefficients* were recognized only many years later as a fundamental breakthrough for 3D discretized quantum gravity and will be addressed in some details. Selected — and by no means complete — lists of further applications and interconnections are included.

1. Introduction

Recall first that the (re)coupling theory of many $SU(2)$ angular momenta — framed mathematically in the structure of the Racah–Wigner tensor algebra — is the most exhaustive formalism in dealing with interacting many-angular momenta quantum systems.[1,2] It suffices here to mention the basic work of Wigner, Racah, Fano and others (see the collection of reprints[3] and the Racah memorial volume quoted in Ref. 4). As such it has been over the years a common tool in advanced applications in atomic and molecular physics, nuclear physics as well as in mathematical physics. In the last three decades there has been a deep interest in applying (extensions of) such notions and techniques to the branch of theoretical physics known as topological quantum field theory, as well as in related discretized models for three-dimensional quantum gravity. More recently the same techniques have been employed for establishing a new framework for quantum computing, the so-called "spin network" quantum simulator.[5]

In section 2 the $6j$ is looked at as a geometric tetrahedron — emerging in the Ponzano–Regge view — the basic *magic brick* in constructing three-dimensional quantum geometries of the Regge type, while in section 3 a few remarks on some developments and connection with other fields are briefly reported. Standard definitions and results on the Wigner $6j$ symbol and its asymptotics are collected in Appendices.

This contribution is largely based on the review paper.[6]

2. The 6j symbol and 3D quantum gravity

From a historical viewpoint the Ponzano–Regge asymptotic formula for the $6j$ symbol,[4] reproduced in (14) of Appendix A.1, together with the seminal paper[7] in which Regge calculus was founded, are no doubt at the basis of all discretized approaches to general relativity, both at the classical and at the quantum level.

In Regge's approach the edge lengths of a triangulated spacetime are taken as discrete counterparts of the metric and a Regge spacetime is a piecewise linear (PL) manifold of dimension D dissected into *simplices*, namely triangles in $D = 2$, tetrahedra in $D = 3$, 4-simplices in $D = 4$ and so on. Inside each simplex either an Euclidean or a Minkowskian metric can be assigned: accordingly, PL manifolds obtained by gluing together D-dimensional simplices acquire an overall PL metric of Riemannian or Lorentzian signature.[a]
Consider a particular triangulation $\mathcal{T}^D(\ell) \to \mathcal{M}^D$, where $\mathcal{M}^D$ is a closed, locally Euclidean manifold of fixed topology and ℓ denotes collectively the (finite) set of edge lengths of the simplices in $\mathcal{T}^D$. The Regge action is given explicitly by (units are chosen such that the Newton constant G is equal to 1)

$$S(\mathcal{T}^D(\ell)) \equiv S^D(\ell) = \sum_{\sigma_i} \mathrm{Vol}^{(D-2)}(\sigma_i)\, \epsilon_i\,, \tag{1}$$

where the sum is over $(D-2)$-dimensional simplices $\sigma_i \in \mathcal{T}^D$ (called hinges or "bones"), $\mathrm{Vol}^{(D-2)}(\sigma_i)$ are their $(D-2)$-dimensional volumes expressed in terms of the edge lengths and ϵ_i represent the deficit angles at σ_i. The latter are defined, for each i, as $2\pi - \sum_k \theta_{i,k}$, where $\theta_{i,k}$ are the dihedral angles between pairs of $(D-1)$-simplices meeting at σ_i and labeled by some k. Thus a positive [negative or null] value of the deficit angle ϵ_i corresponds to a positive [negative or null] curvature to be assigned to the bone i, detected for instance by moving a D-vector along a closed path around the bone i and measuring the angle of rotation. Even such a sketchy description of Regge geometry should make it clear that a discretized spacetime is flat (zero curvature) inside each D-simplex, while the curvature is concentrated at the bones which represent singular subspaces. It can be proven that (under suitable technical conditions) the limit of the Regge action (1) when the edge lengths become smaller and smaller gives the usual Einstein–Hilbert action for a smooth spacetime. Regge equations — the discretized analog of vacuum Einstein's field equations — can be derived from the classical action by varying it with respect to the dynamical variables, *i.e.* the set $\{\ell\}$ of edge lengths of $\mathcal{T}^D(\ell)$, according to Hamilton principle of classical field theory (refer to[8] for a bibliography and brief review on Regge calculus from its beginning up to the 1990's and to the contribution by Ruth M. Williams in the present volume).

[a]Einstein's General Relativity corresponds to the physically significant case of a four-dimensional spacetime endowed with a smooth Lorentzian metric. However, models formulated in non-physical dimensions such as $D = 2, 3$ turn out to be highly non-trivial and very useful in a variety of applications, ranging from conformal field theories and associated statistical models in $D = 2$ to the study of geometric topology of 3-manifolds. Moreover, the most commonly used quantization procedure of such theories has a chance of being well-defined only when the underlying geometry is (locally) Euclidean, see further remarks below.

Recall that Regge calculus gave rise in the early 1980's to a novel approach to quantization of general relativity known as simplicial quantum gravity (see Refs. 8–10 and references therein). The quantization procedure most commonly adopted is the Euclidean path-sum approach, namely a discretized version of Feynman's path-integral describing D-dimensional Regge geometries undergoing "quantum fluctuations" (in Wheeler's words a "sum over histories",[11] formalized for gravity in the so-called Hawking–Hartle prescription[12]). Without entering into technical details, the discretized path-sum approach turns out to be very useful in addressing a number of conceptual open questions in the approach relying on the geometry of smooth spacetimes, although the most significant improvements have been achieved for the $D = 3$ case, which we are going to address in some details in the rest of this section.

Coming to the interpretation of the Ponzano–Regge asymptotic formula for the $6j$ symbol given in (14) of Appendix A.1, we realize that it represents the semiclassical functional, namely the semiclassical limit of a path-sum over all quantum fluctuations, to be associated with the simplest three-dimensional spacetime, an Euclidean tetrahedron T. In fact the (positive frequency part of) argument in the exponential reproduces the Regge action $S^3(\ell)$ for T since in the present case $(D-2)$ simplices are one-dimensional (edges) and $\mathrm{Vol}^{(D-2)}(\sigma_i)$ in (1) are looked at as the associated edge lengths, see the introductory part of Appendix A.

More in general, we denote by $\mathcal{T}^3(j) \to \mathcal{M}^3$ a particular triangulation of a closed three-dimensional Regge manifold $\mathcal{M}^3$ (of fixed topology) obtained by assigning $SU(2)$ spin variables $\{j\}$ to the edges of $\mathcal{T}^3$. The assignment must satisfy a number of conditions, better illustrated if we introduce the *state functional* associated with $\mathcal{T}^3(j)$, namely

$$\mathbf{Z}[\mathcal{T}^3(j) \to \mathcal{M}^3; L] = \Lambda(L)^{-N_0} \prod_{A=1}^{N_1} (-1)^{2j_A} \mathrm{w}_A \prod_{B=1}^{N_3} \phi_B \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}_B \tag{2}$$

where N_0, N_1, N_3 are the number of vertices, edges and tetrahedra in $\mathcal{T}^3(j)$, $\Lambda(L) = 4L^3/3C$ (L is a fixed length and C an arbitrary constant), $\mathrm{w}_A \doteq (2j_A + 1)$ are the dimensions of irreducible representations of $SU(2)$ which weigh the edges, $\phi_B = (-1)^{\sum_{p=1}^{6} j_p}$ and $\{:::\}_B$ are $6j$ symbols to be associated with the tetrahedra of the triangulation. Finally, the Ponzano–Regge *state sum* is obtained by summing over triangulations corresponding to all assignments of spin variables $\{j\}$ bounded by the cut-off L

$$\mathbf{Z}_{PR}[\mathcal{M}^3] = \lim_{L\to\infty} \sum_{\{j\}\le L} \mathbf{Z}[\mathcal{T}^3(j) \to \mathcal{M}^3; L]\,, \tag{3}$$

where the cut-off is formally removed by taking the limit in front of the sum.

3. Remarks and outcomes

It is impossible to review in short, or even mention, the huge number of implications, outcomes and further improvements of the Ponzano–Regge state sum functional (3), as well as its deep and somehow surprising relationships with so many different issues in modern theoretical physics and in pure mathematics. We are going to present in the rest of this section a limited number of items and a few references, whose selection is made mainly on the basis of the longstanding interest in spin networks of the author and her collaborators.

It is worth to stress however the crucial role of this model in the so-called Loop approach to quantum gravity, see[13,14] and references therein, and also the older review paper[10]).

(a) As implicitly stated in,[4] the state sum $\mathbf{Z}_{PR}[\mathcal{M}^3]$ is a topological invariant of the manifold $\mathcal{M}^3$, owing to the fact that its value is actually independent of the particular triangulation, namely does not change under suitable combinatorial transformations. Remarkably, these "moves" are expressed algebraically in terms of the relations given in Appendix A.2, namely the Biedenharn–Elliott identity (17) — representing the moves (2 *tetrahedra*) $\leftrightarrow$ (3 *tetrahedra*) — and of both the Biedenharn–Elliott identity and the orthogonality conditions (18) for $6j$ symbols, which represent the barycentric move together its inverse, namely (1 *tetrahedra*) $\leftrightarrow$ (4 *tetrahedra*).

(b) In[15] a "regularized" version of (3) — based on representation theory of a quantum deformation of the group $SU(2)$ — was proposed and shown to be a well-defined *quantum invariant* for closed 3-manifolds.[b]
Its expression reads

$$\mathbf{Z}_{TV}[\mathcal{M}^3; q] = \sum_{\{j\}} \mathrm{w}^{-N_0} \prod_{A=1}^{N_1} \mathrm{w}_A \prod_{B=1}^{N_3} \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix}_B , \tag{4}$$

where the summation is over all $\{j\}$ labeling highest weight irreducible representations of $SU(2)_q$ ($q = \exp\{2\pi i/r\}$, with $\{j = 0, 1/2, 1\ldots, r-1\}$), $\mathrm{w}_A \doteq (-1)^{2j_A}[2j_A+1]_q$ where $[\,]_q$ denote a quantum integer, $\mathrm{w} = 2r/(q-q^{-1})^2$ and $|\;:::\;|_B$ represents here the q-$6j$ symbol whose entries are the angular momenta $j_i, i = 1, \ldots, 6$ associated with tetrahedron B. If the deformation parameter q is set to 1 one gets $\mathbf{Z}_{TV}[\mathcal{M}^3; 1] = \mathbf{Z}_{PR}[\mathcal{M}^3]$.
It is worth noting that the q-Racah polynomial — associated with the q-$6j$ by a procedure that matches with what can be done in the $SU(2)$ case, see (16) in Appendix A.2 — stands at the top of Askey's q-hierarchy collecting orthogonal q-polynomials of one discrete or continuous variable. On the other hand, the discovery of the Turaev–Viro invariant has provided major developments in the branch of mathematics known as geometric topology.[16] It is still a very active field of research.

(c) The Turaev–Viro or Ponzano–Regge state sums can be generalized in many directions. For instance, they can be extended to simplicial 3-manifold endowed with a two-dimensional boundary,[17] to D-manifolds[18] (giving rise to topological invariants related to suitable discretized topological quantum field theory of the Schwarz type and to not semi-simple Lie groups, just to mention a few.

(d) The fact that the Turaev–Viro state sum is a topological invariant of the underlying (closed) 3-manifold reflects a crucial physical property of gravity in dimension 3 which makes it different from the $D = 4$ case. Loosely speaking, the gravitational field does not possess local degrees of freedom in $D = 3$, and thus any quantized functional can depend only on global features of the manifold encoded into its overall topology. Actually the invariant (4) can be shown to be equal to the square of the modulus of the Witten–Reshetikhin–Turaev invariant, which in turn represents a quantum path-integral of an $SU(2)$ Chern–Simons topological field theory — whose classical action can be shown to be equivalent to Einstein–Hilbert action[19] — written for a closed oriented manifold $\mathcal{M}^3$.[20,21] Then there exists a correspondence

$$\mathbf{Z}_{TV}[\mathcal{M}^3; q] \longleftrightarrow |\,\mathbf{Z}_{WRT}[\mathcal{M}^3; k]\,|^2 , \tag{5}$$

where the level k of the Chern–Simons functional is related to the deformation parameter q of the quantum group.

[b]The adjective quantum refers here to "deformations" of semi-simple Lie groups introduced by the Russian School of theoretical physics in the 1980's in connection with inverse scattering theory. From the mathematical viewpoint the Turaev–Viro invariant, unlike the Ponzano–Regge state sum functional, is always finite and has been evaluated explicitly for some classes of 3-manifolds.

Despite the "topological" nature of the Turaev–Viro (Ponzano–Regge) state sum and Witten–Reshetikhin–Turaev functionals in the case of closed 3-manifolds, whenever a $2D$-dimensional boundary occurs in $\mathcal{M}^3$, giving rise to a pair $(\mathcal{M}^3, \Sigma)$, where Σ is an oriented surface (or possibly the disjoint union of a finite number of surfaces), things change radically. For instance, if we add a boundary to the manifold in Witten–Reshetikhin–Turaev quantum functional, the theory induced on Σ is a Wess–Zumino–Witten (WZW)-type conformal field theory (CFT),[19] endowed with non-trivial quantum degrees of freedom. In particular, the frameworks outlined above can be exploit to establish a direct correspondence between $2D$ Regge triangulations and punctured Riemann surfaces, thus providing a novel characterization of the WZW model on triangulated surfaces on any genus[22] at a fixed level k. We cannot enter here into many technical details on developments on this topic, and refer to the monograph[23] for an upgraded view on the geometry of polytopes and their moduli spaces.

(e) In[24] a $(2+1)$-dimensional decomposition of Euclidean gravity (which takes into account the correspondence (5)) is shown to be equivalent, under mild topological assumptions, to a Gaussian $2D$ fermionic system, whose partition function takes into account the underlying $3D$ topology. More precisely, the partition function for free fermions propagating along "knotted loops" inside a three-dimensional sphere corresponds to a $3D$ Ising model on so-called knot-graph lattices. On the other hand, the formal expression of the $3D$ Ising partition function for a dimer covering of the underlying graph lattice can be shown to coincide with the permanent of the generalized incidence matrix of the lattice.[25,26] Recall first that the permanent of an $n \times n$ matrix A is given by

$$\mathrm{per}[A] = \sum_{\sigma \in \mathrm{S}_n} \prod_{i=1}^{n} a_{i,\sigma(i)} \tag{6}$$

where $a_{i,\sigma(i)}$ are minors of the matrix, $\sigma(i)$ is a permutation of the index $i = 1, 2, \ldots, n$ and S_n is the symmetric group on n elements. A graph lattice $\mathfrak{G}$ associated with a fixed orientable surfaces Σ of genus g embedded in S^3 may be constructed by resorting to the so-called surgery link-presentation. Then the incidence matrix of such piecewise linear graph with, say, $\mathfrak{n}$ vertices, is defined as an $\mathfrak{n} \times \mathfrak{n}$ matrix $A = (a_{ij})$ with entries in $(1, 0)$ according to whether vertices i, j are connected by an edge or not. Finally, the Ising partition function turns out to be a weighted sum — over all possible configurations of knot-graph lattices — of suitable determinants of generalized forms of the incidence matrices which take into account the topology of the underlying manifold. We skip however other technical details and refer to[27,28] for a discussion of algorithmic questions related in turn with item (f) below.

Note finally that the deep relationship between $3D$ quantum field theories that share a topological nature and (solvable) lattice models in $2D$, sketched in the last item by resorting to a specific example, was indeed predicted in the pioneering paper by E. Witten.[29] Not so surprisingly, the basic quantum functional that realizes this connection was identified there with *the expectation value of a certain tetrahedral configuration of braided Wilson lines*, where Wilson lines are quantum observables associated with particle trajectories that in general look like sheafs of braided strands propagating from a surface Σ_1 to another Σ_2, both embedded in a $3D$ background.

(f) The model for universal quantum computation proposed in Ref. 5, the spin network simulator, is based on the (re)coupling theory of $SU(2)$ angular momenta as formulated in the basic texts[1,2] on the quantum theory of angular momentum and the Racah–Wigner algebra respectively. At the first glance the spin network simulator can be thought of as a non-Boolean generalization of the Boolean *quantum circuit model*[c,30] with finite-dimensional, binary coupled computational Hilbert spaces associated with N mutually commuting angular momentum operators and unitary gates expressed in terms of:
i) recoupling coefficients ($3nj$ symbols) between inequivalent binary coupling schemes of $N = (n+1)$ $SU(2)$-angular momentum variables (j-gates);
ii) Wigner rotations in the eigenspace of the total angular momentum $\mathbf{J}$ (M-gates).

The spin network simulator is actually the discretized counterpart of the so-called topological approach to quantum computing developed in Ref. 31 (based, by the way, on the Witten–Reshetikhin–Turaev approach quoted in item (d)).
The role of the model in quantum circuit theory is addressed in Ref. 32.

A few years ago, in collaboration with S. Garnerone and M. Rasetti, we have developed, by resorting to q-deformed quantum automata, a new approach to dealing with classes of algorithmic problems that classically admit only exponential time algorithms. The problems in question arise in the physical context of $3D$ topological quantum field theories discussed above in the light of the fundamental result relating a topological invariant of knots, the Jones polynomial,[33] with a quantum observable given by the vacuum expectation value of a Wilson loop operator[34] associated with closed knotted curves in the Witten–Reshetikhin–Turaev background model.
Without entering into technical details, efficient (polynomial time) quantum algorithms for approximating (with an error that can be made as small as desired) generalizations of Jones polynomial have been found in,[35,36] while the case of topological invariants of 3-manifolds

[c]Recall that this scheme is the quantum version of the classical Boolean circuit in which strings of the basic binary alphabet $(0,1)$ are replaced by collections of qubits, namely quantum states in $(\mathbb{C}^2)^{\otimes N}$, and the gates are unitary transformations that can be expressed, similarly to what happens in the classical case, as suitable sequences of elementary gates associated with the Boolean logic operations *and, or, not.*

has been addressed in.[37] The relevance in having solved this kind of problems stems from the fact that an approximation of the Jones polynomial is sufficient to simulate any polynomial quantum computation.[38]

Appendix A: the Wigner 6j symbol and its symmetries

Given three angular momentum operators $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ — associated with three kinematically independent quantum systems — the Wigner-coupled Hilbert space of the composite system is an eigenstate of the total angular momentum

$$\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 \doteq \mathbf{J} \tag{7}$$

and of its projection J_z along the quantization axis. The degeneracy can be completely removed by considering binary coupling schemes such as $(\mathbf{J}_1 + \mathbf{J}_2) + \mathbf{J}_3$ and $\mathbf{J}_1 + (\mathbf{J}_2 + \mathbf{J}_3)$, and by introducing intermediate angular momentum operators defined by

$$(\mathbf{J}_1 + \mathbf{J}_2) = \mathbf{J}_{12}; \;\; \mathbf{J}_{12} + \mathbf{J}_3 = \mathbf{J} \tag{8}$$

and

$$(\mathbf{J}_2 + \mathbf{J}_3) = \mathbf{J}_{23}; \;\; \mathbf{J}_1 + \mathbf{J}_{23} = \mathbf{J}, \tag{9}$$

respectively. In Dirac notation the simultaneous eigenspaces of the two complete sets of commuting operators are spanned by basis vectors

$$|j_1 j_2 j_{12} j_3;\, jm\rangle \;\text{ and }\; |j_1 j_2 j_3 j_{23};\, jm\rangle, \tag{10}$$

where j_1, j_2, j_3 denote eigenvalues of the corresponding operators, j is the eigenvalue of $\mathbf{J}$ and m is the total magnetic quantum number with range $-j \leq m \leq j$ in integer steps. Note that j_1, j_2, j_3 run over $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ (labels of $SU(2)$ irreducible representations), while $|j_1 - j_2| \leq j_{12} \leq j_1 + j_2$ and $|j_2 - j_3| \leq j_{23} \leq j_2 + j_3$ (all quantum numbers are in $\hbar$ units).

The Wigner $6j$ symbol expresses the transformation between the two schemes (8) and (9), namely

$$|j_1 j_2 j_{12} j_3;\, jm\rangle = \sum_{j_{23}} [(2j_{12}+1)(2j_{23}+1)]^{1/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} |j_1 j_2 j_3 j_{23};\, jm\rangle \tag{11}$$

apart from a phase factor.[d] It follows that the quantum mechanical probability

$$P = [(2j_{12}+1)(2j_{23}+1)] \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}^2 \tag{12}$$

represents the probability that a system prepared in a state of the coupling scheme (8), where j_1, j_2, j_3, j_{12}, j have definite magnitudes, will be measured to be in a state of the coupling scheme (9).

[d]Actually this expression should contain the Racah W-coefficient $W(j_1 j_2 j_3 j; j_{12} j_{23})$ which differs from the $6j$ by the factor $(-)^{j_1+j_2+j_3+j}$. Recall that $(2j_{12}+1)$ and $(2j_{23}+1)$ are the dimensions of the representations labeled by j_{12} and j_{23}, respectively.

The $6j$ symbol may be written as sums of products of four Clebsch–Gordan coefficients or their symmetric counterparts, the Wigner $3j$ symbols. The relations between $6j$ and $3j$ symbols are given explicitly by (see *e.g.* Ref. 39)

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \sum (-)^{\Phi} \begin{pmatrix} a & b & c \\ \alpha & \beta & -\gamma \end{pmatrix} \begin{pmatrix} a & e & f \\ \alpha & \epsilon & -\varphi \end{pmatrix} \begin{pmatrix} d & b & f \\ -\delta & \beta & \varphi \end{pmatrix} \begin{pmatrix} d & e & c \\ \delta & -\epsilon & \gamma \end{pmatrix} \tag{13}$$

where $\Phi = d + e + f + \delta + \epsilon + \varphi$. Here Latin letters stand for j-type labels (integer or half-integers non-negative numbers) while Greek letters denote the associated magnetic quantum numbers (each varying in integer steps between $-j$ and j, $j \in \{a, b, c, d, e, f\}$). The sum is over all possible values of α, β, γ, δ, ϵ, φ with only three summation indices being independent.

On the basis of the above decomposition it can be shown that the $6j$ symbol is invariant under any permutation of its columns or under interchange the upper and lower arguments in each of any two columns. These algebraic relations involve $3! \times 4 = 24$ different $6j$ with the same value and are referred to as *classical symmetries* as opposite to *Regge symmetries* to be discussed in A.2. On the geometric side the classical symmetries of the $6j$ symbol encode the *tetrahedral symmetry* since each $3j$ (or Clebsch–Gordan) coefficient vanishes unless its j-type entries satisfy the triangular conditions, namely $|b - c| \le a \le b + c$, *etc.* This suggests that each of the four $3j$'s in (13) can be be associated with either a 3-valent vertex or a triangle. Here we adopt the Wigner–Ponzano–Regge three-dimensional picture used in Ref. 4, rather than Yutsis' dual representation as a complete graph on four vertices.[40] Accordingly, the $6j$ can be thought of as a real solid tetrahedron T with edge lengths $\ell_1 = a + \frac{1}{2}, \ell_2 = b + \frac{1}{2}$, $\ldots, \ell_6 = f + \frac{1}{2}$ in $\hbar$ units[e] and triangular faces associated with the triads (abc), (aef), (dbf), (dec). This implies in particular that the quantities $q_1 = a + b + c$, $q_2 = a + e + f$, $q_3 = b + d + f$, $q_4 = c + d + e$ (sums of the edge lengths of each face), $p_1 = a + b + d + e$, $p_2 = a + c + d + f$, $p_3 = b + c + e + f$ are all integer with $p_h \ge q_k$ ($h = 1, 2, 3$, $k = 1, 2, 3, 4$). The conditions addressed so far are in general sufficient to guarantee the existence of a non-vanishing $6j$ symbol, but they are not enough to ensure the existence of a geometric tetrahedron T living in Euclidean 3-space with the given edges. More precisely, T exists in this sense if (*and only if*, see the discussion in the introduction of Ref. 4) its square volume $V(T)^2 \equiv V^2$, evaluated by means of the Cayley–Menger determinant, is positive.

A.1 The Ponzano–Regge asymptotic formula

The Ponzano–Regge asymptotic formula for the $6j$ symbol reads[4]

$$\begin{Bmatrix} a & b & d \\ c & f & e \end{Bmatrix} \sim \frac{1}{\sqrt{24\pi V}} \exp\left\{ i \left(\sum_{r=1}^{6} \ell_r \, \theta_r + \frac{\pi}{4} \right) \right\} \tag{14}$$

where the limit is taken for all entries $\gg 1$ (recall that $\hbar = 1$) and $\ell_r \equiv j_r + 1/2$ with $\{j_r\} = \{a, b, c, d, e, f\}$. V is the Euclidean volume of the tetrahedron T and θ_r is the angle between the outer normals to the faces which share the edge ℓ_r. From a quantum mechanical viewpoint, this above probability amplitude has the form of a semiclassical (wave) function

[e] The $\frac{1}{2}$-shift is shown to be crucial in the analysis developed in:[4] for high quantum numbers the length $[j(j+1)]^{1/2}$ of an angular momentum vector is closer to $j + \frac{1}{2}$ in the semiclassical limit.

since the factor $1/\sqrt{24\pi V}$ is slowly varying with respect to the spin variables while the exponential is a rapidly oscillating dynamical phase. Such kind of asymptotic behavior complies with Wigner's semiclassical estimate for the probability, namely $\left\{ \begin{smallmatrix} a & b & d \\ c & f & e \end{smallmatrix} \right\}^2 \sim 1/12\pi V$, to be compared with the quantum probability given in (12). Moreover, according to Feynman's path sum interpretation of quantum mechanics, the argument of the exponential in (14) must represent a classical action, and indeed it can be read as $\sum \mathsf{p}\dot{\mathsf{q}}$ for pairs (p, q) of canonical variables (angular momenta and conjugate angles). Such an interpretation has been improved recently by resorting to multidimensional WKB theory for integrable systems and geometric quantization methods.[41]

A.2 The Racah hypergeometric polynomial

The generalized hypergeometric series, denoted by ${}_pF_q$, is defined on p real or complex numerator parameters $a_1, a_2, \ldots, a_p$, q real or complex denominator parameters $b_1, b_2, \ldots, b_q$ and a single variable z by

$${}_pF_q \left(\begin{matrix} a_1 \ \ldots \ a_p \\ \\ b_1 \ \ldots \ b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_p)_n} \frac{z^n}{n!}, \tag{15}$$

where $(a)_n = a(a+1)(a+2) \cdots (a+n-1)$ denotes a rising factorial with $(a)_0 = 1$. If one of the numerator parameter is a negative integer, as actually happens in the following formula, the series terminates and the function is a polynomial in z.

The key expression for relating the $6j$ symbol to hypergeometric functions is given by the well-known Racah sum rule (see *e.g.*, Ref. 2 topic 11 and Ref. 39, Ch. 9 also for the original references). The final form of the so-called *Racah polynomial* can be recasted in terms of the ${}_4F_3$ hypergeometric function evaluated at $z = 1$ according to

$$\begin{Bmatrix} a & b & d \\ c & f & e \end{Bmatrix} = \Delta(abe)\, \Delta(cde)\, \Delta(acf)\, \Delta(bdf)\, (-)^{\beta_1} (\beta_1 + 1)!$$

$$\times \frac{{}_4F_3 \left(\begin{matrix} \alpha_1 - \beta_1 & \alpha_2 - \beta_1 & \alpha_3 - \beta_1 & \alpha_4 - \beta_1 \\ -\beta_1 - 1 & \beta_2 - \beta_1 + 1 & \beta_3 - \beta_1 + 1 & \end{matrix} ; 1 \right)}{(\beta_2 - \beta_1)!(\beta_3 - \beta_1)!(\beta_1 - \alpha_1)!(\beta_1 - \alpha_2)!(\beta_1 - \beta_3)!(\beta_1 - \alpha_4)!}, \tag{16}$$

where

$$\beta_1 = \min(a + b + c + d; a + d + e + f; b + c + e + f)$$

and the parameters β_2, β_3 are identified in either way with the pair remaining in the 3-tuple $(a + b + c + d; a + d + e + f; b + c + e + f)$ after deleting β_1. The four α's may be identified with any permutation of $(a + b + e; c + d + e; a + c + f; b + d + f)$. Finally, the Δ-factors in front of ${}_4F_3$ are defined, for any triad (abc) as

$$\Delta\,(abc) = \left[\frac{(a + b - c)!(a - b + c)!(-a + b + c)!}{(a + b + c + 1)!} \right]^{1/2}.$$

Such a seemly complicated notation is indeed the most convenient for the purpose of listing further interesting properties of the Wigner $6j$ symbol.

- The Racah polynomial is placed at the top of the Askey hierarchy including all of hypergeometric orthogonal polynomials of one (discrete or continuous) variable.[42] Most commonly encountered families of special functions in quantum mechanics are obtained from the Racah polynomial by applying suitable limiting procedures, as reviewed in.[43] Such an unified scheme provides in a straightforward way the algebraic *defining relations* of the Wigner $6j$ symbol viewed as an orthogonal polynomial of one discrete variable, *cfr.* (16). By resorting to standard notation from the quantum theory of angular momentum, the defining relations are: the Biedenharn–Elliott identity ($R = a+b+c+d+e+f+p+q+r$):

$$\sum_x (-)^{R+x}\,(2x+1)\begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix}\begin{Bmatrix} c & d & x \\ e & f & q \end{Bmatrix}\begin{Bmatrix} e & f & x \\ b & a & r \end{Bmatrix} = \begin{Bmatrix} p & q & r \\ e & a & d \end{Bmatrix}\begin{Bmatrix} p & q & r \\ f & b & c \end{Bmatrix}; \tag{17}$$

the orthogonality relation (δ is the Kronecker delta)

$$\sum_x (2x+1)\begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix}\begin{Bmatrix} c & d & x \\ a & b & q \end{Bmatrix} = \frac{\delta_{pq}}{(2p+1)}. \tag{18}$$

- Given the relation (16), the unexpected new symmetry of the $6j$ symbol discovered by Regge[44] (see also[1,39]) is recognized as a set of permutations on the parameters α, β that leaves ${}_4F_3$ invariant. Combining the Regge symmetry and the classical, tetrahedral ones, one get a total number of 144 algebraic symmetries for the $6j$. Implications of Regge symmetry on the the study of the geometry of the quantum tetrahedron and its semiclassical Hamiltonian dynamics have been extensively addressed in.[45,46]
- The Askey hierarchy of orthogonal polynomials can be extended to a q-hierarchy,[42] on the top of which the q-${}_4F_3$ polynomial stands. Recall that when dealing with quantum invariants of knots and 3-manifolds formulated in the framework of unitary quantum field theory, the deformation parameter q must be a complex root of unity, the case $q = 1$ being considered as the trivial one. We refer to[47,48] for accounts on the theory of q-special functions and q-tensor algebras.

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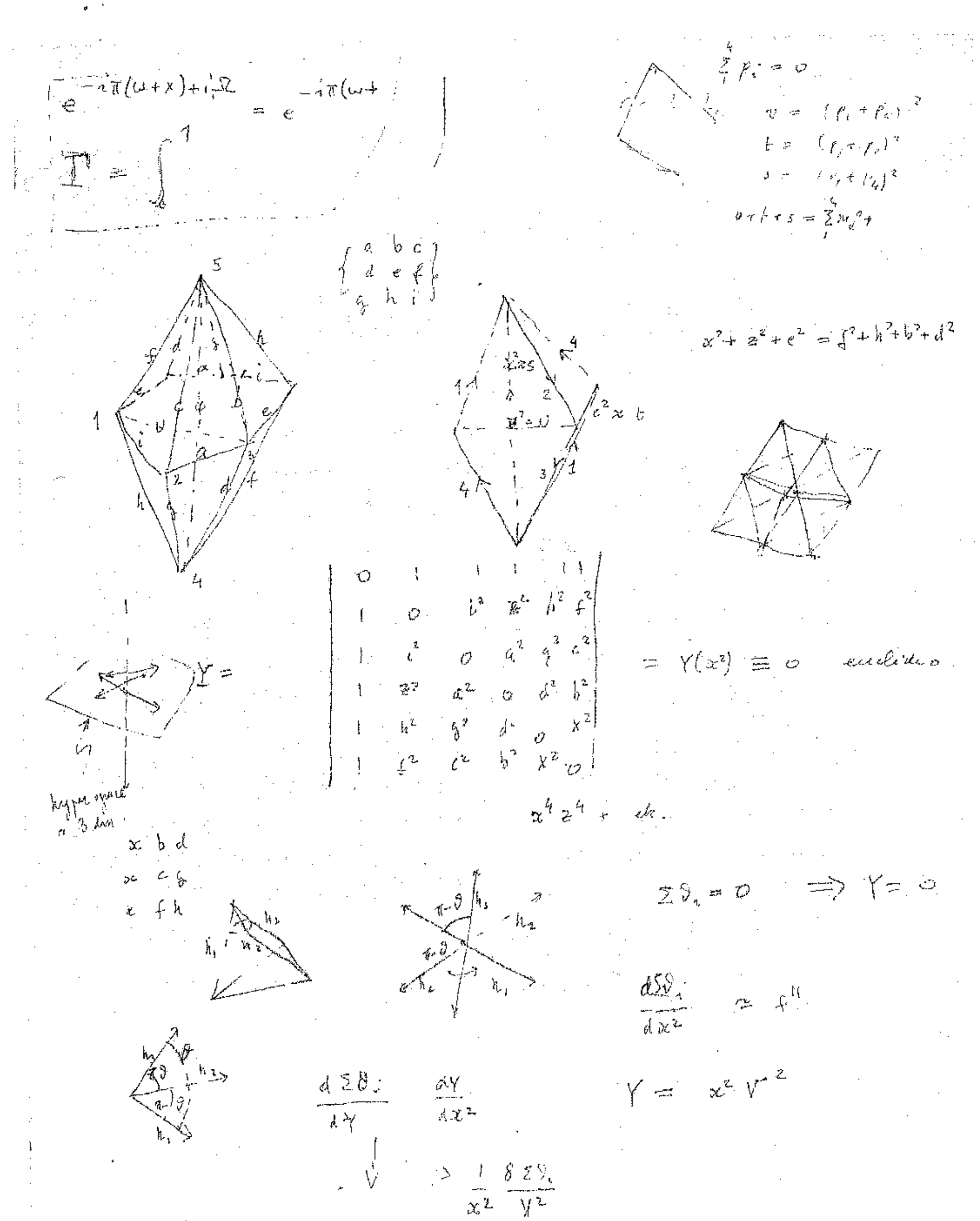

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